On the connectivity of complex affine hypersurfaces, II

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Let $f \in \mathbb{C}[x_0, x_1, ..., x_n]$ be a polynomial and $\mathbf{w} = (w_0, ..., w_n)$ a system of positive integer weights. Consider $f = f_0 + ... + f_e + f_d$ the decomposition of f as a sum of weighted homogeneous components f_i where $deg(f_i) = i$ with respect to $\mathbf{w}$. Here f_d is the top degree nonzero component and we assume that $f_j = 0$ for $e < j < d$ for some integer $e > 0$.

For any polynomial g we denote by ∂g its gradient

$$\partial g = (g_{x_0}, ..., g_{x_n})$$

where g_{x_i} is the partial derivative of g with respect to x_i.

Define the subset $S(f)$ in $\mathbb{C}^{n+1}$ given by the equations $\partial f_d = 0, f_e = 0$. With this notation our result is the following.

THEOREM

(i) Let $\Sigma(f) = \{x \in \mathbb{C}^{n+1}; \partial f = 0\}$ be the set of singular points of the polynomial f. Then $\dim\Sigma(f) \leq \dim S(f)$.

(ii) Any fiber of the polynomial f is $(n - 1 - \dim S(f))$-connected.

Improving previous results on the connectivity of affine hypersurfaces, the first author has proved the part (ii) of this result in the case of the usual weights $\mathbf{w} = (1, 1, ..., 1)$ in [D1].

The proof there was based on the properties of tame and quasi-tame polynomials introduced by Broughton [B], Némethi [N1], [N2] and Némethi-Zaharia [NZ]. More precisely, the general case was reduced to these special classes of polynomials by applying an affine Lefschetz hyperplane section theorem due to Hamm [H]. This proof cannot be applied for arbitrary weights since in general we don't have enough weighted hyperplanes to proceed by induction (even though Lemma 9. in [D1] can be easily extended to show that the polynomial f is quasitame when $\dim S(f) = 0$).

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On the other hand, Némethi and Sabbah [NS] have recently studied tame polynomials defined on affine varieties, but their definition does not seem easily adaptable to keep track of the chosen weights.

Our proof below is based on a powerful general result by Hamm, namely Prop.3 in [H] (which is the key step in his proof of the affine Lefschetz theorem on hyperplane sections and which we slightly correct) and on basic properties of isolated complete intersection singularities, for which we refer the reader to Looijenga's book [L].

The following example shows the strength of the new result over the old.

Example 1.

Let $f : \mathbb{C}^6 \to \mathbb{C}$ be the polynomial given by

$$f(x) = x_0^3 x_1^3 x_2^3 + (x_0 + x_1 + x_2)^7 + x_3^5 + x_4^4 + x_5^3 + (x_0 + x_1 + x_2 + x_3 + x_4 + x_5)^2$$

If we consider the usual weights $\mathbf{w} = (1, 1, ..., 1)$, then $\dim S(f) = 4$, and hence we get that the fibers of f are 0-connected, i.e. they are connected. In this case $d = 9$, $e = 7$

Now, if we use the system of weights $\mathbf{w} = (20, 20, 20, 36, 45, 60)$ chosen such that the top degree form f_d contains as many monomials as possible, then $\dim S(f) = 1$ hence the fibers of f are in fact 3-connected. In this case $d = 180$, $e = 140$.

It is interesting to notice that the topological result above has some useful algebraic consequences expressed in terms of various complexes of differential forms associated naturally to the polynomial f.

To state them, let $A^* = (\Omega^*,\mathrm{d})$ denote the De Rham complex of global regular differential forms on $\mathbb{C}^{n+1}$ with d the exterior differentiation acting on forms (not to be confused to the degree d which occured above).

The first complex associated to f is the complex $K_f^* = (\Omega^*,\mathrm{d}f\wedge)$ which can be identified to the Koszul complex of the partial derivatives of f in the polynomial ring $\mathbb{C}[x_0, ...x_n] = \Omega^0$.

The De Rham complex A^* has a natural subcomplex $B_f = (\mathrm{d}f \wedge \Omega^*,\mathrm{d})$ and a natural quotient complex $C_f = A^*/B_f^*$ called the complex of global relative differential forms.

Finally, one can consider as in Dimca-Saito [DS], the mixture of De Rham and Koszul complexes, namely the complex $D_f = (\Omega^*, \mathrm{d}-\mathrm{d}f\wedge)$.

Corollary.

Let f be a polynomial as above and assume that $k =\dim S(f) < n$. Then

(i) $H^i(K_f^*) = 0$ for all $0 \leq i \leq n - k$;

(ii) $H^1(B_f^*) = \mathbb{C}[f]\mathrm{d}f$ and $H^i(B_f^*) = 0$ for all $i \neq 1, 0 \leq i \leq n - k$;

(iii) $H^0(C_f^*) = \mathbb{C}[f]$ and $H^i(C_f^*) = 0$ for all $0 < i \leq n - k - 1$;

(iv) $H^i(D_f^*) = 0$ for all $0 \leq i \leq n - k$.

Proof of the Theorem

Let N be any common multiple of the weights w_i and consider the integers $m_i = N/w_i$. Let $\phi : \mathbb{C}^{n+1} \to \mathbb{C}^{2n+2}$ be the embedding

$$\phi(x) = (x_0, ..., x_n, x_0^{m_0}, ..., x_n^{m_n})$$

Let $X = f^{-1}(0)$. It is enough to prove that

(i') $\dim X_{sing} \leq \dim S(f)$;

(ii') X is $(n - 1 - \dim S(f))$ connected.

Let $Y = \phi(X)$ and note that $\phi : X \to Y$ is an isomorphism. Let $k = \dim S(f)$ and let $H_1 : l_1 = 0, ..., H_k : l_k = 0$ be generic hyperplanes in $\mathbb{C}^{2n+2}$.

Using repeatedly Lefschetz hyperplane section theorem in [H] we get that the inclusion $Y_0 = Y \cap H_1 ... \cap H_k \to Y$ is an $(n - k)$-equivalence.

Let $X_0 = \phi^{-1}(Y_0)$ and $g_i(x) = l_i(\phi(x))$ be the corresponding pull-back polynomials on $\mathbb{C}^{n+1}$. Note that the weighted homogeneous component $g_{i,N}$ is a polynomial of Pham-Brieskorn type, i.e. a sum $\sum a_{ij} x_j^{m_j}$.

It is easy to see by general transversality arguments involving the parameter space of all the coefficients a_{ij} that we can choose the generic hyperplanes H_i such that one has the following properties:

(a) $A = \{x \in \mathbb{C}^{n+1}; g_1(x) = ... = g_k(x) = 0\}$ is a smooth complete intersection of dimension $n + 1 - k$;

(b) the set $\{x \in \mathbb{C}^{n+1}; x \in S(f), g_{1,N}(x) = ... = g_{k,N}(x) = 0\}$ is just the origin;

(c) there exist weighted homogeneous polynomials $h_1, ..., h_n$ such that the germs at the origin of the sets $\{x \in \mathbb{C}^{n+1}; g_{1,N}(x) = ... = g_{k,N}(x) = f_d(x) = h_1(x) = ... = h_j(x) = 0\}$ is an isolated complete intersection singularity of dimension $\max(n - k - j, 0)$ for all $j = 1, ..., n$.

Note also that we can arrange such that

$(Y_0)_{sing} = Y_{sing} \cap H_1 \cap ... \cap H_k$ and $\dim(Y_0)_{sing}=\dim Y_{sing} - k$

In particular, (i') will follow from the next result.

Lemma 1.

X_0 has at most isolated singularities.

Proof of Lemma 1.

If X_0 has not isolated singularities, then an application of the curve selection lemma, see Milnor [M], would produce a path $p : (0, \epsilon) \to (X_0)_{sing}$ given by a Laurent power series

$$p(t) = c_0 t^s + c_1 t^{s+1} + ...$$

where $c_i \in \mathbb{C}^{n+1}$, $c_0 \neq 0$ and the integer s is strictly negative.

Note that we can reparametrize this path by replacing t with t^N, i. e. we can suppose that all exponents in $p(t)$ with non zero coefficients are divisible by our weights w_i.

There is a standard $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ associated with these weights, namely

$$u * x = (u^{w_0} x_0, ..., u^{w_n} x_n)$$

We can rewrite our path using this action in the form

$$p(t) = t^s * c_0 + t^{s+1} * c_1 + ...$$

where the coefficients c_i's and the exponent s are different in general from those considered first above, but which enjoy the same properties, i.e. $c_0 \neq 0$ and $s < 0$.

Next we have $g_i(p(t)) = g_{i,N}(c_0)t^{sN} + ... = 0$ where the dots represent higher order terms, hence we have $g_{i,N}(c_0) = 0$. In a similar way we get $\partial f_d(c_0) = 0$.

We have also

$0 = df(p(t)) - \sum_{i=0,n} w_i f_{x_i}(p(t))p_i(t) = (d-e)f_e(p(t)) + ... = (d-e)f_e(c_0)t^{se} + ...$
which implies that $f_e(c_0) = 0$, a contradiction with the property (b) above.

This ends the proof of the Lemma and of part (i) in our Theorem.

To continue the proof, we would like to apply Proposition 3. in Hamm [H], and this statement has to be slightly corrected. Indeed, with the notation from [H], consider the following example.

Example 2.

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Let A be $\mathbb{C}^2$ with the trivial Whitney stratification, $f_1 = x^2 y + x$ and $f_2 = x$. Then all the assumptions in Proposition 3. in [H] are fulfilled, but not the conclusion, i.e. $\mathbb{C}^2$ cannot be obtained from the non connected space $f_1^{-1}(0)$ by adjoining 2-cells.

However, looking at Hamm's proof of Proposition 3., it is clear that one needs the following additional condition.

$(\star)$ the set $\{z \in A; |f_1(z)| \leq a_1, ..., |f_N(z)| \leq a_N\}$ is compact for any positive numbers a_j, $j = 1, ..., N$.

This condition allows one to use the Morse theory for the function $|f_1|^2$ to increase a_1 for some fixed (and very large) $a_2, ..., a_N$ as in Hamm's proof. Note that this condition $(\star)$ is not fulfilled in our Example 2.

To prove the claim (ii') we use this completed version of Proposition 3. in [H] for the following data:

The set A in [H] is here the smooth complete intersection $A = \{x \in \mathbb{C}^{n+1}; g_1(x) = ... = g_k(x) = 0\}$, see property (a) above, and hence it satisfies the necessary local connectivity conditions. We choose the trivial Whitney stratification, in which there is just one stratum, namely A.

The functions f_i which appear in [H] are chosen as follows $f_1 = f$ and $f_j = h_{j-1}$ for $j = 2, ..., n+1$.

The conditions these functions have to satisfy are the following.

(c1) the connected components of the critical set of the function $f_1 : A \to \mathbb{C}$ are compact.

This condition is fulfilled, since the same computations as in our Lemma above shows that these components are actually points.

(cj) Let F be the critical set of the function $(f_1, ..., f_j) : A \to \mathbb{C}^j$ for some $j > 1$. Then it is enough to show that $(f_1, ..., f_{j-1}) : F \to \mathbb{C}^{j-1}$ is a proper mapping.

In our case, a critical point for $(f_1, ..., f_j) : A \to \mathbb{C}^j$ is just a singular point of the variety Z given by the equations

$$g_1(x) = ... = g_k(x) = 0, f(x) = a_1, h_1(x) = a_2, ..., h_{j-1}(x) = a_j$$

This variety corresponds to a fiber in a deformation of the isolated complete intersection singularity

$$g_{1,N}(x) = ... = g_{k,N}(x) = f_d(x) = h_1(x) = ... = h_{j-1}(x) = 0$$

For details on how an affine variety can be "localized" to become a fiber in a singularity deformation see [D2], p. 157 and p. 161 (the case discussed there is for the usual weights $\mathbf{w} = (1, ..., 1)$ but the same idea works for general weights).

Hence any such variety Z is either smooth or has just isolated singularities. Projecting on the space $\mathbb{C}^{j-1}$ means that now we consider a family of fibers as above corresponding to a line in the base of the deformation of our singularity. If h_j is chosen general enough, this line is not contained in the discriminant Δ of this deformation, i.e. there is just a finite number of singular fibers when we vary a_j. This follows from the basic fact that a nonconstant regular function on variety with isolated singularities has just finitely many critical values.

It follows that the map $(f_1, ..., f_{j-1}) : F \to \mathbb{C}^{j-1}$ is the composition of two finite maps, the first from F to the discriminant Δ of the deformation and the second the projection $\Delta \to \mathbb{C}^{j-1}$. This explains why the condition (cj) holds as well in our setting for $j > 1$.

Note also that when $j > n+1-k$, then $F = A$ and the map $(f_1, ..., f_{j-1}) : A \to \mathbb{C}^{j-1}$ is a finite map. This shows in particular that our additional condition $(\star)$ holds in our case.

By Proposition 3 in [H] we have that the inclusion $X_0 \to A$ in an $(n-k)$-equivalence. As above, we can identify A with a smooth fiber (Milnor fiber) in the deformation of the corresponding isolated complete intersection, hence A is $(n-k)$-connected being a bouquet of $(n-k+1)$-spheres.

Combining this with the $(n-k)$-equivalence $X_0 \to X$ obtained at the beginning, we have that X is $(n-k-1)$-connected.

This ends the proof of our Theorem.

Proof of the Corollary

The first claim (i) depends only on part (i) of our Theorem. Indeed, the cohomology groups of the Koszul complex are finitely generated $\mathbb{C}[x_0, ..., x_n]$-modules and to prove that one of them is trivial it is enough to show that all its localizations at maximal ideals are trivial.

To check this local property we can use GAGA and replace algebraic localization by analytic localization. At this level the result follows from a general result in Looijenga's book [L], namely Corollary (8.16), p. 157 (take X a smooth germ and $k = 1$ in that statement).

To prove (ii) and (iii), we consider the exact sequence of complexes

$$0 \to B_f^* \to A^* \to C_f^* \to 0$$

This shows that it is enough to prove (ii). We have in fact the following more precise result. By convention, the dimension of the empty set is taken to be -1.

Lemma 2.

Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial function whose critical locus $\Sigma(f)$ has dimension $k < n$. Then the following statements are equivalent:

(i) $H^1(B_f) = \mathbb{C}[f]\mathrm{d}f$ and $H^i(B_f) = 0$ for $i \neq 1, i \leq n - k$;

(ii) the reduced cohomology groups $\tilde{H}^i(F_t, \mathbb{C})$ are trivial for $0 \leq i \leq n - k - 1$.

Proof of Lemma 2.

This proof follows closely Section 9. in Sabbah [S] and for this reason we just mention the main new points.

The algebraic Gauss-Manin system of f is represented (up to a shift) by the complex of $A_1 = \mathbb{C}[t] < \partial >$-modules $GM_f^* = (\Omega^*[\partial], d_f)$ where the $\mathbb{C}$-linear differential d_f is defined by $d_f(\omega\partial^m) = \mathrm{d}\omega\partial^m - \mathrm{d}f \wedge \omega\partial^{m+1}$.

As in Prop. (9.2) in [S], it follows that the condition (ii) is equivalent to the condition

(ii') $H^1(GM_f^*) = \mathbb{C}[f]\mathrm{d}f$ and $H^i(GM_f^*) = 0$ for $i \neq 1, i \leq n - k$.

The complex GM_f^* comes equipped with a decreasing filtration given by

$$F^s GM_f^m = \Omega^m[\partial]_{\leq m-s}$$

where the filtration on the RHS is by the degree with respect to ∂ (Sabbah prefers to work with a similar but increasing filtration in [S]).

The general theory of spectral sequences, see if necessary [Mc], associates to this decreasing, exhaustive and bounded below filtration a spectral sequence with $E_1^{s,t} = H^{s+t}(Gr_F^s GM_f^*)$ converging to $H^{s+t}(GM_f^*)$.

For $t > 0$, we have $E_1^{s,t} = H^{s+t}(K_f^*)$ and hence in particular in our case we have $E_1^{s,t} = 0$ for all $t > 0, s + t \leq n - k$ by our Corollary (i) above.

Moreover, the terms $E_1^{s,0}$ with the corresponding differential $d_1 : E_1^{s,0} \to E_1^{s+1,0}$ coming from the spectral sequence can be identified for $s < n - k$ (since we need again Corollary (i)) to the corresponding initial part in the complex B_f^*.

Since this part of the spectral sequence clearly degenerates at the E_2-term, i.e. $E_2^{s,0} = E_\infty^{s,0}$ for $s \leq n - k$, we obtain the equivalence $(i) \Leftrightarrow (ii)'$ which completes the proof of Lemma 2.

To end the proof of our Corollary, we have just to use the main results in [DS], saying that the cohomology of the complex D_f twisted by -1 is just the reduced cohomology of the general fiber of the polynomial f.

Remarks.

(I) The condition $k = dim\Sigma(f) < n$ is equivalent to saying that the fibers of f are all reduced. This condition is not needed for part (i) of our Corollary (which holds even when $k = n$, but it is necessary for parts (ii)-(iv) since it implies that the general fiber of f is connected.

Indeed, it is well known that any polynomial f can be written as a composition $f(x) = h(g(x))$ where $g : \mathbb{C}^{n+1} \to \mathbb{C}$ has a connected general fiber and $h : \mathbb{C} \to \mathbb{C}$, both g and h being polynomials. Since it is difficult to locate a reference for this fact, here is a short proof of it suggested to the first author by Zaidenberg, to whom we are grateful.

Assume that f is not a constant polynomial and let $\tilde{f} : X \to \mathbb{P}^1$ be a smooth compactification of f. Then the Stein Factorization Theorem, see [Har], p.280, gives a smooth curve C and morphisms $\tilde{g} : X \to C$ and $\tilde{h} : C \to \mathbb{P}^1$ such that $\tilde{h} \circ \tilde{g} = \tilde{f}$ and such that all the fibers of $\tilde{g}$ are connected.

A generic line L in $\mathbb{C}^{n+1}$ has the following properties:

(i) f is not constant when restricted to L;

(ii) the closure $\tilde{L}$ of L in X is a smooth rational curve which meets $\tilde{f}^{-1}(\infty)$ at exactly one point and this intersection is transverse.

Then $\tilde{L}$ is a rational curve and $\tilde{g}|\tilde{L}$ is a non constant map. This implies that $C = \mathbb{P}^1$. Moreover, we have $\tilde{h}^{-1}(\infty) = \infty$, otherwise the condition (ii) above is contradicted. This implies that $\tilde{h}$ give by restriction to $\mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$ a map $h : \mathbb{C} \to \mathbb{C}$ such that $f = h \circ g$ where $g = \tilde{g}|\mathbb{C}^{n+1}$ has its general fiber connected.

(II) For a given polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$ and a given system of weights $\mathbf{w}$ we define the connectivity order of f with respect to $\mathbf{w}$ to be the integer

$$c_{\mathbf{w}}(f) = n - 1 - k$$

where $k = \dim S(f)$.

Then it is easy to see that for the sum of two polynomials $f + g$ where $g : \mathbb{C}^{m+1} \to \mathbb{C}$ and for any weights $\mathbf{w}'$ associated to g such that the corresponding degrees d for f and g coincide (by multiplying any weights $\mathbf{w}$ for f and $\mathbf{w}'$ for g by suitable positive integers this can be always achieved) we have

$$c_{\mathbf{w}, \mathbf{w}'}(f + g) \geq c_{\mathbf{w}}(f) + c_{\mathbf{w}'}(g) + 1$$

in spite of the fact that in general the corresponding degrees e will be different.

Note that the general fiber of $f + g$ is the join of the general fibers of f and g, see [N3], but nothing is known about the special fibers.

(III) Siersma and Tibăr have shown in [ST] that in the case of the usual weights $\mathbf{w} = (1, 1, ..., 1)$ and if $e = d - 1$ and $\dim\Sigma(f) <\dim S(f) = k$, then the general fiber of f is $(n - k)$-connected, i.e. a better by 1 estimate than that given by our Theorem. They also give an example showing that this better estimate fails for the special fibers.

Improving the results in [ST], Tibăr has recently obtained very general connectivity results for the fibers of polynomial mappings, see [T], Thm . (5.5). However the effectiveness of his results depends on the explicit construction of fiber-compactifying extensions of the polynomial function f having a small critical set at infinity. It seems to us that such constructions are difficult to handle for general weights $\mathbf{w}$ or when $e < d - 1$.

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